Starter

1) M is directly proportional to L^3.

When $L = 2$, $M = 160$

Find the value of M when $L = 3$

3) D is proportional to S^2.

$D = 900$ when $S = 20$

Calculate the value of D when $S = 25$

5) The time, T seconds, for a hot sphere to cool is proportional to the square root of the surface area, A m^2, of the sphere.

When $A = 100$, $T = 30$.

Find the value of T when $A = 60$.
Give your answer correct to 3 significant figures.

F Exponentials and logarithms

F2

Know that the gradient of e^{kx} is equal to ke^{kx} and hence understand why the exponential model is suitable in many applications.

Notes

At AS, students should **know** that the gradient of Ae^{kx} is proportional to the value of the function.

At AS, students are **not** expected to differentiate functions involving e^{kx}

This is an unusual situation where the gradient is expected to be known without differentiation being formally used.

Students should understand that the exponential model is suitable in many applications because, if $y = e^{kx}$, $\frac{dy}{dx} = ky$ i.e. the rate of change of y with respect to x is proportional to y.

F Exponentials and logarithms

F7	Understand and use exponential growth and decay; use in modelling (examples may include the use of $\mathbf{e}$ in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use given conditions to determine the values of unknown constant terms in equations of the forms $y = Ae^{bx} + C$ or $P = Ak^t + C$
- form and use exponential equations to make predictions.

5.3 Exponential Models

If then .

Hence .

So the gradient of is ***proportional*** to at the point where is the ***constant of proportionality***.

This property of proportionality occurs frequently in nature and allows us to create mathematical models of events such as radioactive decay and population growth.

5.3 Exponential Models

For example, if the rate of increase of a population of bacteria is directly proportional to the number of bacteria, , then .

As the rate of change of is proportional to , an exponential function is a good model for this situation.

5.3 Exponential Models

An equation of the form

gives an exponential model where A and k are constants.

5.3 Exponential Models

Example 1a

The population, P hundreds of cells, of an organism grows exponentially over time, t hours, according to

a) Find the value of A, complete the table and sketch a graph to represent the data

t	**0**	**5**	**10**	**15**
P	4			

5.3 Exponential Models

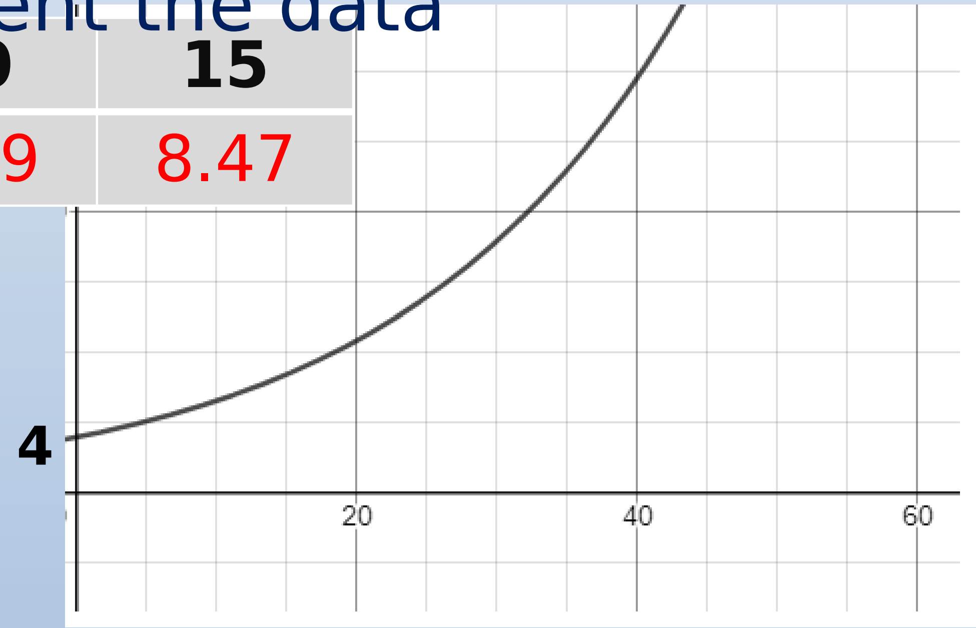
Example 1a

The population, P hundreds of cells, of an organism grows exponentially over time, t hours, according to

a) Find the value of A, complete the table and sketch a graph to represent the data

t	0	5	10	15
P	4	5.14	6.59	8.47

$$P = 4e^{\frac{1}{20}t}$$



Note: do not plot the negative t-axis as time cannot be negative!

5.3 Exponential Models

Example 1b

The population, P hundreds of cells, of an organism grows exponentially over time, t hours, according to

b) What is the rate of increase in the population when t = 5?

When , (3sf)

25.7 cells per hour

5.3 Exponential Models

Example 2a

The growth rate of some bacteria is , where t is the time in hours.

a) How many bacteria are there at the start?

$$y = 500e^{0} = 500$$

5.3 Exponential Models

Example 2b

The growth rate of some bacteria is where t is the time in hours.

b) How many bacteria will there be at this time tomorrow?

$$y = 500e^{48} = 3.51 \times 10^{23}$$

5.3 Exponential Models

Example 2c

The growth rate of some bacteria is where t is the time in hours.

c) How long will it be before there are 1,000,000 bacteria?

hours, to 3sf

Or 3 hours and 48 minutes.

Press this button to convert to hours and minutes!

5.3 Exponential Models

Example 3a

On 1 January 1900, a sculpture was valued at £80

When the sculpture was sold on 1 January 1956, its value was £5000

The value, £V, of the sculpture is modelled by the formula $V = Ak^t$, where t is the time in years since 1 January 1900 and A and k are constants.

(a) Write down the value of A

Sub in :

5.3 Exponential Models

Example 3b

On 1 January 1900, a sculpture was valued at £80

When the sculpture was sold on 1 January 1956, its value was £5000

The value, £V, of the sculpture is modelled by the formula $V = Ak^t$, where t is the time in years since 1 January 1900 and A and k are constants.

(b) Show that $k = 1.07664$ to five decimal places.

Sub in :

(to 5dp)

5.3 Exponential Models

Example 3ci

On 1 January 1900, a sculpture was valued at £80

When the sculpture was sold on 1 January 1956, its value was £5000

The value, £V, of the sculpture is modelled by the formula $V = Ak^t$, where t is the time in years since 1 January 1900 and A and k are constants.

(c) Use this model to:

(i) show that the value of the sculpture on 1 January 2006 will be greater than £200 000

Sub in :

The value is £200,708 > £200,000

5.3 Exponential Models

Example 3cii

On 1 January 1900, a sculpture was valued at £80

When the sculpture was sold on 1 January 1956, its value was £5000

The value, £V, of the sculpture is modelled by the formula $V = Ak^t$, where t is the time in years since 1 January 1900 and A and k are constants.

(c) Use this model to:

(ii) find the year in which the value of the sculpture will first exceed £800 000

It will exceed £800,000 during 2024.

5.3 Exponential Models

Example 3d

On 1 January 1900, a sculpture was valued at £80

When the sculpture was sold on 1 January 1956, its value was £5000

The value, £V, of the sculpture is modelled by the formula $V = Ak^t$, where t is the time in years since 1 January 1900 and A and k are constants.

(d) Explain whether your answer to (c)(ii) is, in reality, likely to be accurate.

No, as the value cannot continue to grow exponentially.

5.3 Exponential Models

Try this:

The concentration, C mg per litre of a particular drug in a patient's bloodstream t hours after it has been administered is given by the formula $C = C_0 e^{-0.2t}$

(a) Initially a patient is given a dose of 5 mg per litre.

(i) Write down the value of C_0

(ii) Find the concentration of the drug 3 hours after it is administered.

(b) The drug becomes ineffective when the concentration drops below 2 mg per litre

How long does it take for the drug to become ineffective? Give your answer to the nearest minute.

5.3 Exponential Models

The concentration, C mg per litre of a particular drug in a patient's bloodstream t hours after it has been administered is given by the formula $C = C_0 e^{-0.2t}$

(a) Initially a patient is given a dose of 5 mg per litre.

(i) Write down the value of C_0

Sub in

5.3 Exponential Models

The concentration, C mg per litre of a particular drug in a patient's bloodstream t hours after it has been administered is given by the formula $C = C_0 e^{-0.2t}$

(a) Initially a patient is given a dose of 5 mg per litre.

(ii) Find the concentration of the drug 3 hours after it is administered.

Sub in :

2.74mg per litre

5.3 Exponential Models

The concentration, C mg per litre of a particular drug in a patient's bloodstream t hours after it has been administered is given by the formula $C = C_0 e^{-0.2t}$

(b) The drug becomes ineffective when the concentration drops below 2 mg per litre

How long does it take for the drug to become ineffective? Give your answer to the nearest minute.

Press this button to convert to hours and minutes!

hours = 4h 35mins

5.3 Exponential Models

<u>Example 4</u>

An area of fungus, $A cm^2$, grows over t days such that

a) Sketch a graph of A against t by calculating A

t	0	10	20	30
A				

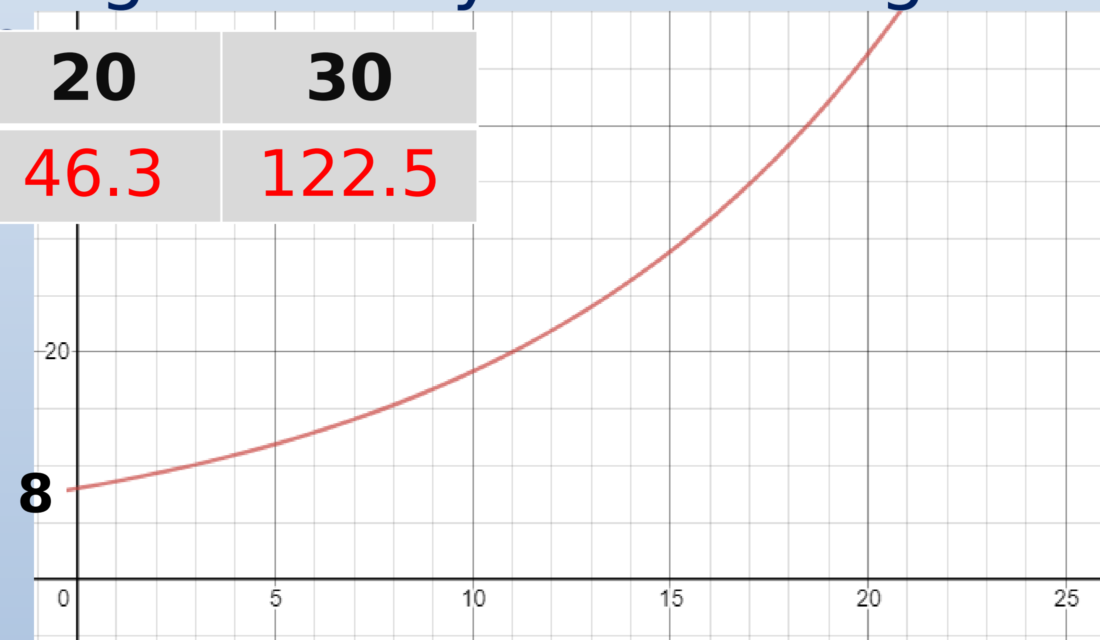
5.3 Exponential Models

Example 4

An area of fungus, $A\text{cm}^2$, grows over t days such that

a) Sketch a graph of A against t by calculating A

t	0	10	20	30
A	8	18.3	46.3	122.5



Asymptote
through
-intercept

5.3 Exponential Models

Example 4

An area of fungus, Acm^2, grows over t days such that

b) How long does it take for the area of the fungus to double?

was 8 at the start so how long to get to 16?

days (3sf)

5.3 Exponential Models

Example 4

An area of fungus, $A\text{cm}^2$, grows over t days such that

c) What is the initial rate of change of A?

When , per day

5.3 Exponential Models

Example 4

An area of fungus, Acm^2, grows over t days such that

d) Why might this model not be realistic for large values of t?

There is no limit placed on the area of fungus – it would increase to infinity according to the model. The model ignores factors such as limited space and changes in conditions.

5.3 Exponential Models

Example 5

A biologist is researching the growth of a certain species of hamster. She proposes that the length, x cm, of a hamster t days after its birth is given by $x = 15 - 12e^{-\frac{t}{14}}$

(a) Use this model to find:

(i) the length of a hamster when it is born.

Sub in :

cm

5.3 Exponential Models

Example 5

A biologist is researching the growth of a certain species of hamster. She proposes that the length, x cm, of a hamster t days after its birth is given by $x = 15 - 12e^{-\frac{t}{14}}$

(a) Use this model to find:

(ii) the length of a hamster after 14 days, giving your answer to three significant figures.

Sub in :

cm (3sf)

5.3 Exponential Models

Example 5

A biologist is researching the growth of a certain species of hamster. She proposes that the length, x cm, of a hamster t days after its birth is given by $x = 15 - 12e^{-\frac{t}{14}}$

(b) (i) Show that the time for a hamster to grow to 10 cm in length is given by

$$t = 14\ln\left(\frac{a}{b}\right)$$

where a and b are integers.

5.3 Exponential Models

Example 5

A biologist is researching the growth of a certain species of hamster. She proposes that the length, x cm, of a hamster t days after its birth is given by $x = 15 - 12e^{-\frac{t}{14}}$

(b) (ii) Find this time to the nearest day.

12 days.